

# DEFORMING NONNORMAL ISOLATED SURFACE SINGULARITIES AND CONSTRUCTING 3-FOLDS WITH $\mathbb{P}^1$ AS EXCEPTIONAL SET

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*To Gert-Martin Greuel on the occasion of his 70th birthday.*

**ABSTRACT.** Normally one assumes isolated surface singularities to be normal. The purpose of this paper is to show that it can be useful to look at nonnormal singularities. By deforming them interesting normal singularities can be constructed, such as isolated, non Cohen-Macaulay threefold singularities. They arise by a small contraction of a smooth rational curve, whose normal bundle has a sufficiently positive subbundle. We study such singularities from their nonnormal general hyperplane section.

## INTRODUCTION

Suppose we are interested in a germ  $(X, 0) \subset (\mathbb{C}^N, 0)$  of a complex space, which has some salient features. Then we would like to describe the singularity  $X$  as explicit as possible. This can be done by giving generators of the local ring  $\mathcal{O}_X$ , or by giving equations for  $X \subset \mathbb{C}^N$ . But in general it is too difficult to do this directly. Instead we first replace the singularity by a simpler one. To recover the original one is then a deformation problem. In a number of situations the simplification process leads to nonnormal singularities.

We formulate the most important simplification process a general principle.

**The hyperplane section principle.** *The (general) hyperplane section of a singularity has a local ring with the same structure as the original singularity, but one embedding dimension lower, and which is much easier to describe.*

A nonnormal surface singularity occurs as general hyperplane of a normal three-dimensional isolated singularity, if this singularity is not Cohen-Macaulay. Such singularities can occur as result of small contractions. In higher dimensions a resolution (with normal crossings exceptional divisor) is in general not the correct tool for understanding the singularity. But it may happen that a small resolution exists, meaning (in dimension three) that the exceptional set is only a curve. The simplest case is that the curve is a smooth rational curve. Nevertheless, such a singularity can be quite complicated, as it need not be Cohen-Macaulay. This happens if the normal bundle of the curve is  $\mathcal{O}(a) \oplus \mathcal{O}(b)$  with  $a > 1$ . One has always that  $2a + b < 0$  [3, 20], and Ando has given examples of the extremal case  $(a, b) = (n, -2n - 1)$  by exhibiting transition functions. We study the contraction of such a curve using the hyperplane section principle.

The first example of a manifold containing an exceptional  $\mathbb{P}^1$  with normal bundle with positive subbundle, namely  $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ , was given by Laufer [16]. Pinkham

gave a construction as total space of a 1-parameter smoothing of a partial resolution of a rational double point [21]. We consider smoothings of partial resolutions of non-rational singularities. In this case the total space does blow down, but not to a smoothing of the original singularity. Instead, the special fibre is a nonnormal surface singularity. We retrieve Ando's examples using the canonical model of the hypersurface singularity  $z^2 = f_{2n+1}(x, y)$ . We also give new examples.

Using SINGULAR [11] it is possible to give explicit equations for some cases. We do not compute deformations of the canonical model, but deformations of the non-normal surface singularity  $X$ . We study in detail the simplest case, where  $X$  differs not too much from its normalisation  $\tilde{X}$ , meaning that  $\delta(X) = \dim \mathcal{O}_{\tilde{X}}/\mathcal{O}_X = 1$ . We take the equation of  $\tilde{X}$  of the form  $z^2 = f(x, y)$ . It turns out that it is in fact possible to give general formulas.

A 1-parameter deformation of a resolution of a normal surface singularity blows down to a deformation of the singularity if and only if the geometric genus is constant. Otherwise the special fibre is nonnormal. Given a 2-dimensional hypersurface singularity, the general singularity with the same resolution graph is not Gorenstein, and not quasi-homogeneous in the case that hypersurface is quasi-homogenous. Again, deforming a nonnormal quasi-homogeneous surface singularity gives a method to find equations for surface singularities with a given star-shaped graph. We give an example using the same general formulas as for small contractions (Example 5).

The structure of this paper is as follows. In the first section we discuss invariants for nonnormal surface singularities. In the next section we compute deformations for a nonnormal model of a surface singularity of multiplicity two. In section 3 we recall in detail the relation between deformations of a (partial) resolution and of the singularity itself. The final section treats  $\mathbb{P}^1$  as exceptional curve, with explicit formulas based on the previous calculations.

## 1. INVARIANTS OF NONNORMAL SINGULARITIES

### 1.1. Normalisation.

**Definition 1.** A reduced ring  $R$  is *normal* if it is integrally closed in its total ring of fractions. For an arbitrary reduced ring  $R$  its *normalisation*  $\bar{R}$  is the integral closure of  $R$  in its total ring of fractions. A singularity  $(X, 0)$  (i.e., the germ of a complex space) is *normal* if its local ring  $\mathcal{O}_{(X,0)}$  is normal. The *normalisation* of a reduced germ  $(X, 0)$  is a multi-germ  $(\bar{X}, \bar{0})$  with semi-local ring  $\mathcal{O}_{(\bar{X}, \bar{0})} = \bar{\mathcal{O}}_{(X,0)}$ . The normalisation map is  $\nu: (\bar{X}, \bar{0}) \rightarrow (X, 0)$ , or in terms of rings  $\nu^*: \mathcal{O}_{(\bar{X}, \bar{0})} \rightarrow \mathcal{O}_{(X,0)}$ .

We have the following function-theoretic characterisation of normality, see e.g. [10, p. 143]. Let  $\Sigma$  be the singular locus of a reduced complex space  $X$  and set  $U = X \setminus \Sigma$ , with  $j: U \rightarrow X$  the inclusion map. Then  $X$  is normal at  $p \in X$  if and only if for arbitrary small neighbourhoods  $V \ni p$  every bounded holomorphic function on  $U \cap V$  has a holomorphic extension to  $X \cap V$ . If  $\text{codim } \Sigma \geq 2$ , then  $\bar{\mathcal{O}}_X = j_*\mathcal{O}_U$ .

**1.2. Cohen-Macaulay singularities.** For a two-dimensional isolated singularity normal is equivalent to Cohen-Macaulay, but in higher dimensions this is no longer true. A local ring is *Cohen-Macaulay* if there is a regular sequence of length equal to the dimension of the ring. A  $d$ -dimensional germ  $(X, 0)$  is Cohen-Macaulay, if its local ring is Cohen-Macaulay. An equivalent condition is that there exists a finite projection  $\pi: (X, 0) \rightarrow (\mathbb{C}^d, 0)$  with fibres of constant multiplicity (i.e., the map  $\pi$  is flat), see e.g., [10, Kap. III § 1]. From both descriptions it follows directly that a singularity is Cohen-Macaulay if and only if its general hyperplane section is so. In particular, a general hyperplane section of a normal but not Cohen-Macaulay isolated 3-fold singularity is not normal.

A cohomological characterisation, in terms of local cohomology, of isolated Cohen-Macaulay singularities is that  $H_{\{0\}}^q(X, \mathcal{O}_X) = 0$  for  $q < d$ . Normality implies only the vanishing for  $q < 2$ . The local cohomology can be computed from a resolution  $\tilde{X} \rightarrow X$ , as  $H^q(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H_{\{0\}}^{q+1}(X, \mathcal{O}_X)$  for  $1 \leq q \leq n - 2$  [12, Prop. 4.2]. If  $\tilde{X}$  is a good resolution of an isolated singularity with exceptional divisor  $E$ , then the map  $H^i(\mathcal{O}_{\tilde{X}}) \rightarrow H^i(\mathcal{O}_E)$  is surjective for all  $i$  [24, Lemma 2.14]. This implies that a 3-fold singularity is not Cohen-Macaulay if the exceptional divisor of a good resolution is an irregular surface  $F$  (meaning that  $q = h^1(\mathcal{O}_F) > 0$ ). The easiest example of such a singularity is the cone over an irregular surface.

**Example 1.** The equations of the known families of smooth irregular surfaces in  $\mathbb{P}^4$  are discussed in [5, Sect. 4]. They admit a large symmetry group, the Heisenberg group. The lowest degree case is that of elliptic quintic scrolls. Their homogeneous coordinate ring has a minimal free resolution of type

$$0 \leftarrow \mathcal{O}_S \leftarrow \mathcal{O}(-3)^5 \xleftarrow{L} \mathcal{O}(-4)^5 \xleftarrow{x} \mathcal{O}(-5) \leftarrow 0 ,$$

where  $L$  is a matrix

$$\begin{pmatrix} 0 & -s_1x_4 & -s_2x_3 & s_2x_2 & s_1x_1 \\ s_1x_2 & 0 & -s_1x_0 & -s_2x_4 & s_2x_3 \\ s_2x_4 & s_1x_3 & 0 & -s_1x_1 & -s_2x_0 \\ -s_2x_1 & s_2x_0 & s_1x_4 & 0 & -s_1x_2 \\ -s_1x_3 & -s_2x_2 & s_2x_1 & s_1x_0 & 0 \end{pmatrix}$$

and  $x$  is the vector  $(x_0, x_1, x_2, x_3, x_4)^t$ . The constants  $(s_1 : s_2)$  are homogeneous coordinates on the modular curve  $X(5) \cong \mathbb{P}^1$ . The  $i$ -th column of the matrix  $\bigwedge^4 L$  is divisible by  $x_i$  and the equations of the scroll are the five resulting cubics, which can be obtained from the following one by cyclic permutation of the indices:

$$s_1^4 x_0 x_2 x_3 - s_1^3 s_2 (x_1 x_2^2 + x_3^2 x_4) - s_1^2 s_2^2 x_0^3 + s_1 s_2^3 (x_1^2 x_3 + x_2 x_4^2) + s_2^4 x_0 x_1 x_4 .$$

A general hyperplane section is a quintic elliptic curve in  $\mathbb{P}^3$ , which is not projectively normal. In fact, the linear system of hyperplane sections is not complete, and therefore the curve is not a (linear) normal curve.

**1.3. The  $\delta$ -invariant.** Let  $(X, 0) \subset (\mathbb{C}^N, 0)$  be an isolated singularity. One measures how far the singularity is from being normal with the  $\delta$ -invariant:

$$\delta(X, 0) = \dim(\mathcal{O}_{\tilde{X}, \nu^{-1}(0)} / \mathcal{O}_{X,0}) .$$

For plane curves this is the familiar  $\delta$ -invariant, which is also called the number of virtual double points. In higher dimensions it is not the correct double point number. One has to consider the double point locus of the composed map  $\varphi: \overline{X} \rightarrow X \rightarrow \mathbb{C}^N$  (see [13] for the general theory of double point schemes). The expected dimension of the double point locus is  $2 \dim X - N$ . As  $X$  has an isolated singularity, necessarily  $N \geq 2 \dim X$ , and the best results are in case of equality.

Consider a map  $\varphi: \overline{X} \rightarrow Y$  with  $\overline{X}$  a complete intersection and  $Y$  smooth of dimension twice the dimension of  $\overline{X}$ . As measure of degeneracy of the map  $\varphi$  Artin and Nagata [4] introduced (following Mumford) half the size of source double point locus :

$$\Delta(\varphi) = \frac{1}{2} \dim \text{Ker} (\mathcal{O}_{\overline{X} \times_Y \overline{X}} \longrightarrow \mathcal{O}_X) .$$

This dimension is stable under deformations of  $\varphi$ , and by deforming to an immersion with only nodes one sees that  $\Delta(\varphi)$  is an integer, as in that case  $\overline{X} \times_Y \overline{X}$  splits into the diagonal and a finite set with free  $\mathbb{Z}/2$ -action.

For plane curve singularities it follows by deforming to a curve with only nodes that  $\delta(X) = \Delta(\varphi)$ , cf. [27, 3.4]. As the image of  $\varphi$  is given by one equation, the images of a family of maps form a flat family and  $\mathcal{O}_{\overline{X}_S}/\varphi^*\mathcal{O}_{Y_S}$  is flat over the base  $S$ . In higher dimensions the images of a family of maps need not form a flat family and  $\delta(X)$  may be larger than  $\Delta(\varphi)$ .

**Example 2** (cf. [4, (5.8)]). Map three copies of  $(\mathbb{C}^2, 0)$  generically to  $(\mathbb{C}^4, 0)$ , say by  $(x, y, z, w) = (s_1, t_1, 0, 0) = (0, 0, s_2, t_2) = (s_3, t_3, s_3, t_3)$ . The image of this map  $\varphi$  lies on the quadric  $xw = yz$ , and there are four more cubic equations. The singularity is rigid, one computes that  $T^1 = 0$ . The  $\delta$ -invariant is equal to 4. On the other hand, the double point number  $\Delta(\varphi)$  is 3, and the general deformation of the map is obtained by moving the third plane. The ideal of the image is then the intersection of the ideals  $(z, w)$ ,  $(x, y)$  and  $(x - z - a, y - w - b)$ . There are eight cubic equations, obtained by multiplying the generators of the three ideals in all possible ways. Specialising to  $a = b = 0$  one obtains the product of the ideals  $(z, w)$ ,  $(x, y)$  and  $(x - z, y - w)$ . This ideal has an embedded component. The same ideal is obtained if one does not consider the image with its reduced structure, but with its Fitting ideal structure, as in [27, §1]; indeed, that construction commutes with base change.

#### 1.4. Simultaneous normalisation.

**Definition 2.** Let  $f: \mathcal{X} \rightarrow S$  be a flat map between complex spaces, such that all fibres are reduced. A *simultaneous normalisation* of  $f$  is a finite map  $\nu: \overline{\mathcal{X}} \rightarrow \mathcal{X}$  such that all fibres of the composed map  $f \circ \nu$  are normal, and that for each  $s \in S$  the induced map on the fibre  $\nu_s: \overline{\mathcal{X}}_s = (f \circ \nu)^{-1}(s) \rightarrow \mathcal{X}_s = f^{-1}(s)$  is the normalisation.

Criteria for the existence of a simultaneous normalisation are given by Chiang-Hsieh and Lipman [6], see also [11, II.2.6]. If the nonnormal locus of  $\mathcal{X}$  is finite over the base  $S$ , and  $S$  is smooth 1-dimensional, then the normalisation of  $\mathcal{X}$  is a simultaneous normalisation if and only if  $\delta(\mathcal{X}_s)$  (defined as the sum over the  $\delta$ -invariants of the singular points) is constant. For families of curves this results

holds over an arbitrary normal base  $S$ , a result originally due to Teissier and Raynaud.

**1.5. The geometric genus.** The geometric genus of a normal surface singularities was introduced by Wagreich [28], using a resolution  $\pi: (\tilde{X}, E) \rightarrow (X, 0)$ , as  $p_g = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ . In terms of cycles on the resolution one has  $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \varprojlim H^1(Z, \mathcal{O}_Z)$ , where  $Z$  runs over all effective divisors with support on the exceptional set. This means that  $p_g$  is the maximal value of  $h^1(\mathcal{O}_Z)$ , see [22, 4.8]. Wagreich also defined the arithmetic genus of the singularity as the maximal value of  $p_a(Z)$ , where  $p_a(Z) = 1 - h^0(\mathcal{O}_Z) + h^1(\mathcal{O}_Z)$ . This is a topological invariant. One computes  $p_a(Z)$  by the adjunction formula:  $p_a(Z) = 1 + \frac{1}{2}Z(Z + K)$ . The geometric genus has an interpretation independent of a resolution, as  $\dim(H^0(U, \Omega_U^2)/L^2(U, \Omega_U^2))$ , where  $L^2(U, \Omega_U^2)$  is the subspace of square-integrable 2-forms on  $U = \tilde{X} \setminus E = X \setminus 0$  [15].

For a not necessarily normal isolated singularity  $(X, 0)$  the geometric genus is a combination of the  $\delta$ -invariant and invariants from the resolution. This makes sense, as the resolution factors over the normalisation. In any dimension we define, following Steenbrink [24, (2.12)]:

**Definition 3.** Let  $(X, 0)$  be an isolated singularity of pure dimension  $n$  with resolution  $(\tilde{X}, E)$ . The *geometric genus* is

$$p_g(X, 0) = -\delta(X, 0) + \sum_{q=1}^{n-1} (-1)^{q-1} \dim H^q(\tilde{X}, \mathcal{O}_{\tilde{X}}).$$

The dimension of  $H^q(\tilde{X}, \mathcal{O}_{\tilde{X}})$  does not depend on the chosen resolution, and is therefore an invariant of the singularity; one way to see this is using an intrinsic characterisation: one has  $H^q(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H_{\{0\}}^{q+1}(X, \mathcal{O}_X)$  for  $1 \leq q \leq n-2$ , and  $H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^0(U, \Omega_U^n)/L^2(U, \Omega_U^n)$  [12, Prop. 4.2]. We remark that  $\delta(X, 0) = \dim H_{\{0\}}^1(X, \mathcal{O}_X)$ . For isolated Cohen-Macaulay singularities all terms except the last one vanish, so  $p_g = (-1)^n \dim H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}})$ , which is the direct generalisation of Wagreich's formula.

By the results of [9] the geometric genus is semicontinuous under deformation. More precisely, let  $\pi: \tilde{X} \rightarrow X$  be a resolution of a pure dimensional space and let the complex  $M_X^\bullet$  be the third vertex of the triangle constructed on the natural map  $i: \mathcal{O}_X \rightarrow R^\bullet \pi_* \mathcal{O}_{\tilde{X}}$ :

$$\begin{array}{ccc} & M_X^\bullet & \\ +1 \swarrow & & \nwarrow \\ \mathcal{O}_X & \longrightarrow & R^\bullet \pi_* \mathcal{O}_{\tilde{X}} \end{array}$$

Then  $M_X^{-1} = \text{Ker } \mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}}$ ,  $M_X^0 = \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ ,  $M_X^i = R^i \pi_* \mathcal{O}_{\tilde{X}}$  for  $0 < i < n = \dim X$  and all other  $M_X^i$  are zero. If  $X$  has isolated singularities define the partial Euler-Poincaré characteristics

$$\psi_i(X) = \sum_{j=0}^{n-i} (-1)^j \dim M_X^{n-j-i-1}, \quad 0 \leq i \leq n.$$

**Proposition 1** ([9, Théorème 1]). *For an equidimensional flat morphism  $f: \mathcal{X} \rightarrow S$  with  $\mathcal{X}$  smooth outside a closed set, finite over the base, the functions  $s \mapsto \psi_i(\mathcal{X}_s)$  are upper semicontinuous.*

This result has the following corollaries, which are relevant for us.

**Corollary 2** ([19, p. 255]). *If a nonnormal reduced isolated surface singularity  $X$  is smoothable, then  $\delta(X) \leq p_g(\tilde{X})$ .*

**Corollary 3** ([14, (14.2)]). *Let  $f: X \rightarrow T$  be a morphism from a normal threefold to the germ of a smooth curve. If  $X_0 = f^{-1}(0)$  has only isolated singularities and the normalisation  $\bar{X}_0$  has only rational singularities, then  $\bar{X}_0 = X_0$ .*

For a 1-parameter deformation of an isolated nonnormal surface singularity with rational normalisation semicontinuity of  $\psi_0 = -\delta$  and  $\psi_1 = \delta$  implies that  $\delta$  is constant. Therefore there is a simultaneous normalisation. The same is not necessarily true for infinitesimal deformations. In the next section we give an example, where there exist obstructed deformations without simultaneous normalisation.

## 2. COMPUTATIONS

In this section we describe equations and deformations for surface singularities with  $\delta = 1$ , whose normalisation is a double point, so given by an equation of the form  $z^2 = f(x, y)$ , with  $f \in \mathfrak{m}^2$ .

We recall the set-up for deformations of singularities (for details see [25]). One starts from a system of generators  $(g_1, \dots, g_k)$  of the ideal of the singularity  $X$ . We also need generators of the module of relations, which we write as matrix  $(r_{ij})$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, l$ . So we have  $l$  relations  $\sum g_i r_{ij} = 0$ . We perturb the generators to  $G_i(x, t)$  with  $G_i(x, 0) = g_i(x)$ . These describe a (flat) deformation of  $X$  if it is possible to lift the relations: there should exist a matrix  $R(x, t)$  with  $R(x, 0) = r(x)$  such that  $\sum G_i R_{ij} = 0$  for all  $j$ . One can take this as definition of flatness. In particular, for an infinitesimal deformation  $G_i(x, \varepsilon) = g_i(x) + \varepsilon g'_i(x)$  (with  $\varepsilon^2 = 0$ ) one needs the existence of a matrix  $r'(x)$  such that  $\sum (g_i + \varepsilon g'_i)(r_{ij} + \varepsilon r'_{ij}) = \varepsilon \sum (g'_i r_{ij} + g_i r'_{ij}) = 0$ , or equivalently that  $\sum g'_i r_{ij}$  lies in the ideal generated by the  $g_i$ . Deformations, induced by coordinate transformations, are considered to be trivial. To find the versal deformation, one takes representatives for all possible non-trivial infinitesimal deformations, and tries to lift to higher order. The obstructions to do this define the base space of the versal deformation.

We consider a subring  $\mathcal{O}$  of  $\bar{\mathcal{O}} = \mathbb{C}\{x, y, z\}/(z^2 - f(x, y))$  with  $\delta = \dim \bar{\mathcal{O}}/\mathcal{O} = 1$ . We need a system of generators for the defining ideal. This, and the possible deformations depend on the subring in question. We write  $\mathcal{O} = \mathbb{C} + L + \mathfrak{m}^2$ , where  $L$  is a two-dimensional subspace of  $\mathfrak{m}/\mathfrak{m}^2$ , which can be given as kernel of a linear form  $l: \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathbb{C}$ ,  $l = ax + by + cz$ .

First suppose that  $c \neq 0$ ; we may assume that  $c = 1$ . Generators of  $\mathcal{O}$  are then

$$(1) \quad \begin{array}{lll} \xi_1 = x - az & \eta_1 = y - bz & \zeta_2 = z^2 \\ \xi_2 = z(x - az) & \eta_2 = z(y - bz) & \zeta_3 = z^3 \end{array}$$

This system of generators is not minimal, as we not yet have taken the relation  $z^2 = f(x, y)$  into account. Because  $f \in \mathfrak{m}^2$ , it can be written in terms of the



generators; for example,  $x^2 = \xi_1^2 + 2az\xi_1 + a^2z^2 = \xi_1^2 + 2a\xi_2 + a^2\zeta_2$ , and  $x^3 = \xi_1^3 + 3a\xi_1\xi_2 + 3a^2\zeta_2\xi_1 + a^3\zeta_3$ . The relations  $z^2 = f(x, y)$  and  $z^3 = zf(x, y)$  lead to equations

$$\begin{aligned}\zeta_2 &= \varphi_2(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3) , \\ \zeta_3 &= \varphi_3(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3) .\end{aligned}$$

Therefore the variables  $\zeta_2$  and  $\zeta_3$  can be eliminated and the embedding dimension of  $\mathcal{O}$  is four. Coordinates are  $\xi_1, \xi_2, \eta_1$  and  $\eta_2$ , and equations for the corresponding singularity  $X$  are

$$(2) \quad \begin{aligned}\xi_1\eta_2 &= \xi_2\eta_1 \\ \xi_2^2 &= \xi_1^2\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2) \\ \xi_2\eta_2 &= \xi_1\eta_1\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2) \\ \eta_2^2 &= \eta_1^2\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2)\end{aligned}$$

where  $\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2)$  is obtained from  $\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3)$  by eliminating  $\zeta_2$  and  $\zeta_3$ .

**Proposition 4.** *The nonnormal singularity with equations (2) has only deformations with simultaneous normalisation.*

*Proof.* We write down the four relations between the generators:

$$\begin{aligned}(\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\xi_1 - (\xi_2^2 - \xi_1^2\varphi_2)\eta_1 - (\xi_1\eta_2 - \xi_2\eta_1)\xi_2 &= 0 \\ (\eta_2^2 - \eta_1^2\varphi_2)\xi_1 - (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\eta_1 - (\xi_1\eta_2 - \xi_2\eta_1)\eta_2 &= 0 \\ (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\xi_2 - (\xi_2^2 - \xi_1^2\varphi_2)\eta_2 - (\xi_1\eta_2 - \xi_2\eta_1)\xi_1\varphi_2 &= 0 \\ (\eta_2^2 - \eta_1^2\varphi_2)\xi_2 - (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\eta_2 - (\xi_1\eta_2 - \xi_2\eta_1)\eta_1\varphi_2 &= 0\end{aligned}$$

All perturbations of the equations lie in the maximal ideal. By using coordinate transformations we can assume that the first equation  $\xi_1\eta_2 - \xi_2\eta_1$  is not perturbed at all. We perturb the other equations as  $\xi_2^2 - \xi_1^2\varphi_2 - \varepsilon_{\xi\xi}$ ,  $\xi_2\eta_2 - \xi_1\eta_1\varphi_2 - \varepsilon_{\xi\eta}$  and  $\eta_2^2 - \eta_1^2\varphi_2 - \varepsilon_{\eta\eta}$ . We get four equations holding in  $\mathcal{O}$ , which can be written as

$$\text{rank} \begin{pmatrix} \xi_2 & \xi_1 & \varepsilon_{\xi\xi} & \varepsilon_{\xi\eta} \\ \eta_2 & \eta_1 & \varepsilon_{\xi\eta} & \varepsilon_{\eta\eta} \end{pmatrix} \leq 1 .$$

The first minor is the equation  $\xi_1\eta_2 - \xi_2\eta_1$ , and the last minor vanishes identically, as we are considering infinitesimal deformations. Thus the perturbations can be written as  $\varepsilon_{\xi\xi} = \xi_1^2\varepsilon_{11} + \xi_1\xi_2\varepsilon_{12}$  (without  $\xi_2^2$ -term, as  $\xi_2^2 = \xi_1^2\varphi_2$ ),  $\varepsilon_{\xi\eta} = \xi_1\eta_1\varepsilon_{11} + \xi_1\eta_2\varepsilon_{12}$  and  $\varepsilon_{\eta\eta} = \eta_1^2\varepsilon_{11} + \eta_1\eta_2\varepsilon_{12}$  with  $\varepsilon_{11}$  and  $\varepsilon_{12}$  the same functions of the variables in all three perturbations. We can arrange that  $\varepsilon_{11}$  only depends on  $\xi_1$  and  $\eta_1$ , and not on  $\xi_2$  and  $\eta_2$ , by collecting terms in  $\varepsilon_{12}$ . The coordinate transformation  $\xi_2 \mapsto \xi_2 + \frac{1}{2}\xi_1\varepsilon_{12}$ ,  $\eta_2 \mapsto \eta_2 + \frac{1}{2}\eta_1\varepsilon_{12}$  gets rid of the terms with  $\varepsilon_{12}$ . The resulting

equations

$$\begin{aligned}\xi_1\eta_2 &= \xi_2\eta_1 \\ \xi_2^2 &= \xi_1^2(\varphi_2 + \varepsilon_{11}) \\ \xi_2\eta_2 &= \xi_1(\varphi_2 + \varepsilon_{11}) \\ \eta_2^2 &= \eta_1^2(\varphi_2 + \varepsilon_{11})\end{aligned}$$

define not only an infinitesimal deformation, but also a genuine deformation. We conclude that the base space of the versal deformation is smooth (even though  $T^2$  is not zero).

The simultaneous normalisation is given by  $z^2 = f + \varepsilon_{11}(x - az, y - bz)$ .  $\square$

To obtain interesting other deformations we have to assume that  $c = 0$ . Then the subspace  $L$  of  $\mathfrak{m}/\mathfrak{m}^2$  is given as kernel of a linear form  $l = ax + by$ . Assuming  $a = 1$  we find  $z$  and  $y - bx$  as generators of degree 1. As we have not yet specified the form of  $f(x, y)$  we can apply a coordinate transformation to achieve that  $b = 0$ . So we take the linear form  $l = x$ . Generators of the ring  $\mathcal{O}$  are now  $z, y, w = zx, v = yx, x_2 = x^2$  and  $x_3 = x^3$ . If none of these monomials occurs in  $f(x, y)$ , then the embedding dimension is 6.

The formulas

$$(3) \quad \begin{array}{lll} x_2 = x^2 & y = y & z = z \\ x_3 = x^3 & v = xy & w = xz \end{array}$$

define an injective map  $\nu: \mathbb{C}^3 \rightarrow \mathbb{C}^6$ . Let  $Y$  be the image, which is an isolated 3-dimensional singularity. As it is not even normal, its nine equations cannot directly be given in determinantal format, but this is possible by allowing some redundancy. Consider the maximal minors of the  $2 \times 6$  matrix

$$(4) \quad \begin{pmatrix} z & y & x_2 & w & v & x_3 \\ w & v & x_3 & zx_2 & yx_2 & x_2^2 \end{pmatrix}.$$

There are three equations occurring twice, like  $vw - zy x_2$ , while the last three are obtained by multiplying the first three by  $x_2$ . Therefore we get nine generators of the ideal. The determinantal format gives relations between the generators, and a computation with SINGULAR [8] shows that there are no other relations. A further computation gives  $\dim T_Y^1 = 1$ . The 1-parameter deformation is given by the maximal minors of

$$(5) \quad \begin{pmatrix} z & y & x_2 & w & v & x_3 \\ w & v & x_3 & z(x_2 + s) & y(x_2 + s) & x_2(x_2 + s) \end{pmatrix}.$$

It comes from deforming the map  $\nu$  to

$$(6) \quad \begin{array}{lll} x_2 = x^2 - s & y = y & z = z \\ x_3 = x(x^2 - s) & v = xy & w = xz \end{array}$$

The singularity of the general fibre is isomorphic to the one-point union of two 3-spaces in 6-space.



Now we restrict the map  $\nu$  to a hypersurface  $\{z^2 = f(x, y)\} \subset \mathbb{C}^3$ , with  $z^2 - f \in \nu^*\mathfrak{m}_6$ . We assume that  $f \in (y^2, yx^2, x^4)$ . We get two additional equations by writing  $z^2 - f$  and  $x(z^2 - f)$  in the coordinates on  $\mathbb{C}^6$ . We write

$$(7) \quad \begin{aligned} z^2 &= y\alpha + x_2\beta, \\ zw &= v\alpha + x_3\beta. \end{aligned}$$

The second equation is obtained from the first by *rolling factors* using the matrix (4), i.e., replacing in each monomial one occurrence of an entry of the upper row by the entry of the lower row in the same column. One can roll once more, to give an expression for  $w^2$ , but as we have the equation  $w^2 = x_2z^2$ , the resulting equation is just the  $z^2$ -equation multiplied by  $x_2$ .

The singularity has a large component with simultaneous normalisation. For this just perturb  $z^2 - f$  with elements of  $\nu^*\mathfrak{m}_6$ . That means that we can write the two additional equations, using rolling factors. This works also for the deformation of  $\nu(\mathbb{C}^3)$  given by the equations (5) and the map (6). But there is also another deformation direction.

**Proposition 5.** *The singularity  $X$  with normalisation of the form  $z^2 = f(x, y)$ , where  $f \in (y^2, yx^2, x^4)$ , and local ring with generators (3) has an infinitesimal deformation, not tangent to the component with simultaneous normalisation.*

*Proof.* The existence is suggested by a SINGULAR [8] computation in examples. The result can be checked by hand.

We first give the relations between the equations. We write  $g_i$  for the equations of  $Y$ , coming from the matrix (4), and  $h_k$  for the two additional equations (7). A relation has the form  $\sum g_i r_{ij} + h_k s_{kj} = 0$ . We can pull it back to  $\mathbb{C}^3$  with the map  $\nu$ . It then reduces to  $(z^2 - f)(s_{1j} + x s_{2j}) = 0$ . Therefore we find six relations, generating all relations with non-zero  $s_{kj}$ . They are found by reading the matrix product

$$\begin{pmatrix} w & -z \\ v & -y \\ x_3 & -x_2 \\ zx_2 & -w \\ yx_2 & -v \\ x_2^2 & -x_3 \end{pmatrix} \begin{pmatrix} z & y & x_2 \\ w & v & x_3 \end{pmatrix} \begin{pmatrix} z \\ -\alpha \\ -\beta \end{pmatrix}$$

in two different ways: the product of the last two matrices is a column vector containing the two equations  $h_1, h_2$ , while the product of the first two is a  $6 \times 3$  matrix, with antisymmetric upper half containing the minors of the middle matrix (the first half of the matrix (4)) and symmetric lower half, containing the remaining six generators. The other relations, with  $s_{kj} = 0$ , are the determinantal relations between the equations  $g_i$  of the three-dimensional singularity  $Y$ .

To find a solution to  $\sum g'_i r_{ij} + h'_k s_{kj} = 0 \in \mathcal{O}_X$  it suffices to compute on the normalisation. We start with the determinantal relations for the  $g_i$ . As the second row of the matrix is just the first one multiplied with  $x \in \overline{\mathcal{O}}_X$ , it suffices to consider

only those relations obtained by doubling the first row. Consider the relation

$$0 = \begin{vmatrix} z & y & x_2 \\ z & y & x_2 \\ w & v & x_3 \end{vmatrix} = (yx_3 - vx_2)z - (zx_3 - wx_2)y + (zv - yw)x_2.$$

The perturbation of the three equations involved, obtained from  $z \cdot z - \alpha \cdot y - \beta \cdot x_2 = 0 \in \overline{\mathcal{O}}_X$ , cannot be extended to the other equations. It is possible to extend after multiplication with  $x \in \overline{\mathcal{O}}_X$ . Let  $\overline{\alpha} \in \mathcal{O}_X$  be the element with  $\overline{\alpha} = x\alpha \in \overline{\mathcal{O}}_X$  and likewise  $\overline{\beta} = x\beta$ . Then  $zw = y\overline{\alpha} + x_2\overline{\beta}$ . We do not perturb the equations  $h_1$  and  $h_2$ , nor the equation  $x_3^2 - x_2^3$ . We solve for the perturbations of the remaining equations, and check that all equations  $\sum g'_i r_{ij} = 0 \in \mathcal{O}_X$  described above are satisfied. The result is the following infinitesimal deformation (written as column vector):

$$G^t = g^t + \varepsilon g^t = \begin{pmatrix} zv - yw \\ zx_3 - wx_2 \\ yx_3 - vx_2 \\ w^2 - x_2z^2 \\ wv - x_2zy \\ v^2 - x_2y^2 \\ wx_3 - zx_2^2 \\ vx_3 - yx_2^2 \\ x_3^2 - x_2^3 \end{pmatrix} + \varepsilon \begin{pmatrix} \overline{\beta} \\ -\overline{\alpha} \\ -w \\ 2z\alpha \\ 2y\alpha + x_2\beta \\ 2zy \\ x_2\alpha \\ zx_2 \\ 0 \end{pmatrix}$$

□

For the extension to higher order one needs further divisibility properties of  $\alpha$  and  $\beta$ . Indeed, if  $p_g(\overline{X}) = 0$ , then by Corollaries 2 and 3 the deformation of the Proposition has to be obstructed.

**Example 3.** Let the normalisation be a rational double point. Specifically, we take  $\overline{X}$  of type  $A_3$ , given by  $z^2 = y^2 + x^4$ . For the nonnormal singularity  $X$  the dimension of  $T^1$  is equal to 8. There is a 7-dimensional component with simultaneous normalisation: six parameters are seen in the equation  $z^2 - y^2 - x_2^2 + a_1z + a_2y + a_3x_2 + a_4w + a_5v + a_6x_3$ , and  $s$  is a parameter for the deformation (6) of the map  $\nu: \mathbb{C}^3 \rightarrow \mathbb{C}^6$ . Finally let  $t$  be the coordinate for infinitesimal deformation of Proposition 5. A computation of the versal deformation with SINGULAR [8, 18] shows that the equations for the base space are  $st = a_1t = a_2t = a_3t = a_4t^2 = a_5t^2 = a_6t^2 = t^3 = 0$ .

For  $\overline{X}$  given by  $z^2 = f(x, y)$  with  $f \in \mathfrak{m}^k$  the structure of the versal deformation stabilises for large  $k$ . Computations in examples with SINGULAR [8] suggest that  $T^2$  always has dimension 16 (this is also true for the singularity of Example 3) and that there are in general 11 equations for the base space. There is one component of codimension 1 with simultaneous normalisation and two other components: a singularity  $z^2 = f(x, y)$  with  $f \in \mathfrak{m}^k$ ,  $k \geq 6$  can be deformed into  $\tilde{E}_7$  or  $\tilde{E}_8$ .

We compute the component related to  $\tilde{E}_7$ . This can be done by determining the versal deformation in negative degrees of the singularity  $z^2 = -ay^4 + bx^5$ , where  $a$  and  $b$  are parameters, using [18]. After a coordinate transformation the equations

of the base space do not depend on the parameters. We find the component. For other singularities we have just to substitute suitable functions of space and deformation variables for the parameters in the formulas we find.

The result is rather complicated, so we do not give all equations, but use

$$G_3 = x_3y - x_2v + tw$$

to eliminate the variable  $w$ . Four of the original equations do not involve  $w$ . They are

$$\begin{aligned} G_6 &= v(v + a_1t^2) - x_2y^2 - 2tzy - bt^2x_3 + a_2t^2y^2 + a_0t^2(x_2 + a_2t^2) - a_3bt^4y, \\ G_8 &= x_3v - x_2^2y - tzx_2 - 2a_4t^2y(y^2 + a_0t^2) \\ &\quad + ba_4t^4(v + a_1t^2) - (a_3v + a_2x_2)t^2y - a_2t^3z - a_3bt^4(x_2 + a_2t^2), \\ G_9 &= x_3^2 - x_2(x_2 + a_2t^2)^2 - a_3t^2(x_2 + a_2t^2)(v + a_1t^2) \\ &\quad + a_4t^2(v + a_1t^2)^2 - 4a_4t^2x_2y^2 - a_3^2t^4y^2, \\ H_1 &= z^2 + a_4(y^2 + a_0t^2)^2 - bx_3x_2 \\ &\quad + (a_3v + a_2x_2)(y^2 + a_0t^2) + a_1x_2v + a_0x_2^2 + a_3bt^3z - a_4b^2t^4x_2. \end{aligned}$$

The ideal with  $w$  eliminated has three more generators, which we give the name of the original generators leading to them.

$$\begin{aligned} G_1 &= x_3(y^2 + a_0t^2) - x_2yv + zt(v + a_1t^2) \\ &\quad - bt^2x_2(x_2 + a_2t^2) + a_3t^2y(y^2 + a_0t^2) + a_1t^2x_2y, \\ G_2 &= x_3x_2y - x_2v(x_2 + a_2t^2) + tzx_3 - a_4t^2(v + a_1t^2)(y^2 + a_0t^2) \\ &\quad + 2a_4bt^4x_2y + a_3a_2t^4(y^2 + a_0t^2) - a_3t^3zy + a_3a_0t^4x_2, \\ H_2 &= x_2z(v + a_1t^2) - x_3yz + a_4t(v + a_1t^2)y(y^2 + a_0t^2) \\ &\quad + a_3tx_2y^3 + a_2tx_2vy + ta_1x_2^2y + ta_0x_3x_2 \\ &\quad - btx_2^2(x_2 + a_2t^2) + a_3t^2zy^2 - 2a_4bt^3x_2y^2 - a_3a_2t^3y(y^2 + a_0t^2). \end{aligned}$$

To obtain the full ideal one has to add the equation used to eliminate  $w$  and saturate with respect to the variable  $t$ .

**Example 4.** We use our equations to write down the deformation in the case that  $z^2 - f$  is a surface singularity of type  $\tilde{E}_7$ . We start from  $z^2 = y^4 - \nu x^2y^2 + x^4$ . There is a second modulus, coming from changing the generator  $y$  to  $y + \lambda x$ . By a coordinate transformation we can keep  $y$  as generator, and take

$$(8) \quad z^2 = y^4 - \mu xy^3 - \nu x^2y^2 + x^4$$

as normalisation. Fixing these moduli the component is one-dimensional. We describe its total space. Its equations are obtained by putting  $b = a_1 = 0$ ,  $a_0 = a_4 = -1$ ,  $a_3 = \mu$  and  $a_2 = \nu$  in the formulas above. The equation  $H_1$  becomes

$$H_1 = z^2 - (y^2 - t^2)^2 - x_2^2 + (\mu v + \nu x_2)(y^2 - t^2).$$

It is reducible if  $y^2 = t^2$ . If  $y^2 \neq t^2$ , the equation  $G_1$  shows that  $x_3$  also can be eliminated. There is one more equation not involving  $x_3$ :

$$G_6 = v^2 - x_2(y^2 + t^2) - 2tzy + \nu t^2(y^2 - t^2).$$

The local ring of the total space is a section ring  $\oplus H^0(V, nL)$  for some ample line bundle on a projective surface  $V$ . The dimension of  $H^0(V, L)$  is two. We look at the normalisation of a general hyperplane section  $t = \lambda y$ . We assume that  $\lambda^2 \neq 1$ , Equation  $G_6$  shows that  $v/y$  is in the normalisation. We set it equal to  $(1 - \lambda^2)x$ .

If  $\lambda^2 + 1 \neq 0$  we can eliminate  $x_2$  and find (after dividing by  $(1 - \lambda^2)^2$ ) that the normalisation is given by

$$(z + 2\lambda t^2 - \nu \lambda y^2)^2 = (\lambda^2 + 1)^2(y^4 - \mu xy^3 - \nu x^2 y^2 + x^4),$$

which for all  $\lambda$  (with  $\lambda^2 + 1 \neq 0$ ) is isomorphic to (8). The sections with  $\lambda^2 = 1$  are reducible. One sees that  $V$  is a ruled surface over the elliptic curve with equation (8), with two sections of self-intersection zero,  $E_1$  and  $E_{-1}$ , and  $L$  is given by the linear system  $|E_1 + f|$ , where  $f$  is a fibre. The general element of the linear system is a section of the ruled surface with self intersection 2.

**Remark 1.** Each nonnormal singularity  $X \subset \mathbb{C}^N$  is the image of its normalisation  $\bar{X}$ , giving rise to a map  $\bar{X} \rightarrow \mathbb{C}^N$ . Not every deformation of this map is flat. To give an example for  $X$  as above with normalisation  $z^2 = f(x, y)$ , we observe that we can deform the map by using the same map  $\mathbb{C}^3 \rightarrow \mathbb{C}^6$  and perturbing the equation arbitrarily, say  $z^2 = f(x, y) + u$ . For flatness of the images one needs to perturb both equations  $z^2 = y\alpha + x_2\beta$ ,  $zw = v\alpha + x_3\beta$  with elements in the local ring of the nonnormal singularity, where the second is obtained from the first by multiplying with  $x$  (on the normalisation). The perturbation  $z^2 = f(x, y) + u$  is not of this type.

### 3. DEFORMATIONS OF A RESOLUTION

A deformation of a resolution of a normal surface singularity blows down to a deformation of the singularity if and only if  $h^1(\mathcal{O}_{\bar{X}})$  is constant [23, 29]. If not, the total space of a 1-parameter deformation still blows down to a three-dimensional singularity, but the special fibre is no longer normal.

Let more generally  $\pi_0: (Y, E) \rightarrow (X, 0)$  be the contraction of an exceptional set  $E$  to a point, with  $(Y, E)$  not necessarily smooth, of dimension  $n$ . In principle  $Y$  is a germ along  $E$ , but we work always with a strictly pseudo-convex representative, which we denote with the same symbol  $Y$ . Then  $\mathcal{O}_X = (\pi_0)_*\mathcal{O}_Y$ . In particular,  $X$  is normal if  $Y$  is normal. Consider now a deformation  $\tilde{f}: \mathcal{Y} \rightarrow S$  of  $Y$  over a reduced base space  $(S, 0)$ . One can assume that  $\tilde{f}$  has a 1-convex representative. All the exceptional sets in all fibres can be contracted: let  $\pi: \mathcal{Y} \rightarrow \mathcal{X}$  be the Remmert reduction, so  $\mathcal{O}_{\mathcal{X}} = \pi_*\mathcal{O}_{\mathcal{Y}}$  with  $\tilde{f} = f \circ \pi$ . Then  $f: \mathcal{X} \rightarrow S$  is a deformation of  $\mathcal{X}_0 := f^{-1}(0)$ . The question is whether  $f$  also is a deformation of  $X$ , i.e., whether  $X \cong \mathcal{X}_0$ . The answer is the following [23, Satz 3], cf. [29] for the algebraic case.

**Theorem 6.** *Let  $\mathcal{Y} \xrightarrow{\pi} \mathcal{X} \xrightarrow{f} S$  be the Remmert reduction of the deformation  $\tilde{f}: \mathcal{Y} \rightarrow S$  of  $Y$ , over a reduced base space  $(S, 0)$ . Then the special fibre  $\mathcal{X}_0$  of  $f: \mathcal{X} \rightarrow S$  is the Remmert reduction of  $Y = \mathcal{Y}_0$  if and only if the restriction map  $H^0(Y, \mathcal{O}_{\mathcal{Y}}) \rightarrow H^0(Y, \mathcal{O}_Y)$  is surjective. This is the case if  $\dim H^1(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s})$  is constant on  $(S, 0)$ .*

The converse of the last clause holds if  $\dim H^2(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s})$  is constant [23, Satz 5]. This is automatically satisfied in the case of interest to us, that  $Y \rightarrow X$  is a modification of a normal surface singularity. The result then says that a deformation of the modification blows down to a deformation of the singularity if and only if  $p_g$  is constant.

In the proof one reduces to the case of a 1-parameter deformation. Let us consider what happens in that case, so we have a diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ & \searrow & \swarrow \\ & T & \end{array}$$

Let  $t$  be the coordinate on  $T$ , and consider multiplication with  $t$  on  $\mathcal{O}_{\mathcal{Y}}$  and  $\mathcal{O}_{\mathcal{X}}$ . We get the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow H^0(\mathcal{O}_{\mathcal{Y}}) & \xrightarrow{t} & H^0(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & H^0(\mathcal{O}_Y) & \longrightarrow & H^1(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{t} H^1(\mathcal{O}_{\mathcal{Y}}) \\ & & \downarrow \cong & & \downarrow & & \\ 0 \rightarrow \mathcal{O}_{\mathcal{X}} & \xrightarrow{t} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_{\mathcal{X}_0} & \longrightarrow & 0 \end{array}$$

It shows that  $\mathcal{O}_{\mathcal{X}_0}$  is equal to  $H^0(\mathcal{O}_Y) = \mathcal{O}_X$  if and only if the restriction map  $H^0(\mathcal{O}_{\mathcal{Y}}) \rightarrow H^0(\mathcal{O}_Y)$  is surjective. If  $\dim H^1(\mathcal{Y}_t, \mathcal{O}_{\mathcal{Y}_t})$  is constant, then  $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$  is a free  $\mathcal{O}_{\mathcal{Y}}$ -module, on which multiplication with  $t$  is injective, so the restriction map  $H^0(\mathcal{O}_{\mathcal{Y}}) \rightarrow H^0(\mathcal{O}_Y)$  is surjective.

To study the converse, and what happens if  $\dim H^1(\mathcal{Y}_t, \mathcal{O}_{\mathcal{Y}_t})$  is not constant, we restrict to the case that  $H^2(\mathcal{O}_{\mathcal{Y}}) = 0$ .

**Proposition 7.** *Let  $\mathcal{Y} \xrightarrow{\pi} \mathcal{X} \xrightarrow{f} T$  be the Remmert reduction of the 1-parameter deformation  $\tilde{f}: \mathcal{Y} \rightarrow T$  of the space  $Y$ , with  $H^2(\mathcal{O}_{\mathcal{Y}}) = 0$ . Then*

$$\dim H^1(\mathcal{Y}_0, \mathcal{O}_{\mathcal{Y}_0}) = \dim H^1(\mathcal{Y}_t, \mathcal{O}_{\mathcal{Y}_t}) - \dim((\pi_0)_* \mathcal{O}_Y / \mathcal{O}_{\mathcal{X}_0}),$$

where  $t \neq 0$ .

*Proof.* The upper line in the commutative diagram extends as

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}_{\mathcal{Y}}) & \xrightarrow{t} H^0(\mathcal{O}_{\mathcal{Y}}) \longrightarrow H^0(\mathcal{O}_Y) \longrightarrow \\ & \longrightarrow H^1(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{t} H^1(\mathcal{O}_{\mathcal{Y}}) \longrightarrow H^1(\mathcal{O}_Y). \end{aligned}$$

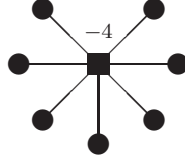
The generic rank of the  $\mathcal{O}_T$ -module  $H^1(\mathcal{O}_{\mathcal{Y}})$  is equal to  $\dim H^1(\mathcal{Y}_t, \mathcal{O}_{\mathcal{Y}_t})$ , but also equal to  $\dim \text{Coker}(\cdot t) - \dim \text{Ker}(\cdot t)$ . This proves the formula.  $\square$

In particular, if  $Y$  is a resolution of the normal surface singularity  $X$ , then the formula of the proposition says that  $p_g(\mathcal{X}_t) = p_g(X) - \delta = p_g(\mathcal{X}_0)$ . Here we use essentially that we have a 1-parameter deformation: over a higher dimensional base space  $\delta$  will be larger than  $p_g(X) - p_g(\mathcal{X}_t)$ : for a 1-parameter curve in the base  $p_g(X) - p_g(\mathcal{X}_t)$  gives how many functions fail to extend, but it will depend on the curve which ones do not extend.

**Remark 2.** If we have a smoothing, or more generally if  $p_g(\mathcal{X}_t) = 0$ , then  $H^1(\mathcal{O}_{\mathcal{Y}})$  is  $t$ -torsion and isomorphic to  $H^1(\mathcal{O}_Y)$ .

The above Proposition gives a way to construct normal surface singularities with the same resolution graph as a given hypersurface singularity, but with lower  $p_g$ . If the drop in  $p_g$  is equal to  $\delta$ , we start from a nonnormal model of the hypersurface singularity with  $\delta$ -invariant  $\delta$  and compute deformations without simultaneous resolution. If the singularity is quasi-homogeneous, deformations of positive weight will have constant topological type of the resolution.

**Example 5.** Consider the hypersurface singularity  $z^2 = x^7 + y^7$  with resolution graph:



This is a singularity with  $p_g = 3$ , but its arithmetic genus is equal to two. The general, non-Gorenstein singularity with the same graph has indeed  $p_g = 2$ . We can use the computations of the previous section. We get a weighted homogeneous deformation by putting  $b = x_2$ ,  $a_0 = a_1 = a_2 = a_3 = 0$  and  $a_4 = -y^3$ . The equation  $G_3 = x_3y - x_2v + tw$  shows that the  $w$ -deformation has positive weight  $-(8-9) = 1$ .

We give the equations for the fibre at  $t = 1$ . Then the equation  $H_2$  lies in the ideal of the other ones and we obtain the following six equations

$$\begin{aligned} G_6 &= v^2 - x_2y^2 - 2zy - x_2x_3 , \\ G_8 &= x_3v - x_2^2y - zx_2 + 2y^6 - x_2y^3v , \\ G_9 &= x_3^2 - x_2^3 - v^2y^3 + 4x_2y^5 , \\ H_1 &= z^2 - y^7 - x_2^2x_3 + y^3x_2^3 . \\ G_1 &= x_3y^2 - x_2yv + zv - x_2^3 , \\ G_2 &= x_3x_2y - x_2^2v + zx_3 + vy^5 - 2x_2^2y^4 . \end{aligned}$$

One checks that this ideal indeed defines a singularity with the above resolution graph by resolving it; one possible method is to blow up a canonical ideal.

#### 4. $\mathbb{P}^1$ AS EXCEPTIONAL SET

In understanding normal surface singularities the resolution is a very important tool. For 3-fold singularities this is not the case for several reasons. First of all, there is no unique minimal resolution. The combinatorics of a good resolution (i.e., the exceptional divisor has normal crossings) seems prohibitive in general. But now there is a new phenomenon, that there may exist resolutions in which the exceptional set is not a divisor, but an analytic set of lower dimension. This is called a small resolution. It means that in a certain sense the singularity is not too singular. For 3-fold singularities we are talking about resolutions with exceptional set a curve.

If the exceptional curve  $C$  is rational, then its normal bundle splits as  $\mathcal{O}(a) \oplus \mathcal{O}(b)$  with  $a \geq b$ . Rather surprisingly, the number  $a$  can be positive. Laufer gave in



[16] an example of a curve  $C \subset \tilde{X}$  with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ , which even contracts to a hypersurface singularity  $X$ .

Generalising earlier results of Ando (see [2]) and Nakayama[20], that contractability limits the value of  $(a, b)$  to  $2a + b < 0$ , Ando proved [3]:

**Theorem 8.** *Let  $C$  be a smooth exceptional curve in an  $m$ -dimensional manifold  $\tilde{X}$ , and let  $M$  be a subbundle of the normal bundle  $N_{C/\tilde{X}}$  of maximal degree  $a$  and put  $b = \deg N_{C/\tilde{X}} - a$ . Then  $2a + b < 0$  and  $a + b < 0$ . Moreover, if  $C$  is rational, then  $a + b \leq 1 - m$ .*

Ando [1, 3] has also existence results. In particular, in dimension 3 he exhibits examples with the maximal normal bundle  $\mathcal{O}(n) \oplus \mathcal{O}(-2n - 1)$ , by giving, in the style of Laufer, transition functions between two copies of  $\mathbb{C}^3$ . The resulting singularity is not Cohen-Macaulay for  $n > 1$ . Consider more generally a rational curve with normal bundle of type  $(a, b)$  with  $a > 1$ . To see that  $H^2_{\{0\}}(X, \mathcal{O}_X) \cong H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$ , let  $\mathcal{J}$  be the ideal sheaf of  $C$  in  $\tilde{X}$  and look at the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_C \longrightarrow 0, \\ 0 \longrightarrow \mathcal{J}^2 \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow 0. \end{aligned}$$

We have a surjection  $H^0(\mathcal{O}_{\tilde{X}}) \twoheadrightarrow H^0(\mathcal{O}_C) = \mathbb{C}$ . As  $C$  is rational,  $H^1(\mathcal{O}_C) = 0$  and therefore  $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(\tilde{X}, \mathcal{J})$ . Because  $C$  is a curve,  $H^2(\tilde{X}, \mathcal{J}^2) = 0$  and we get a surjection  $H^1(\tilde{X}, \mathcal{J}) \twoheadrightarrow H^1(C, \mathcal{J}/\mathcal{J}^2)$ . As  $\mathcal{J}/\mathcal{J}^2$  is the dual of the normal bundle, we have that  $h^1(C, \mathcal{J}/\mathcal{J}^2) = a - 1$  and therefore  $h^1(\mathcal{O}_{\tilde{X}}) \geq a - 1$ .

Pinkham gave a construction for  $C$  with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(-3)$  [21], using smoothings of partial resolutions of rational double points. The easiest example, starting from a  $D_4$ -singularity, is described in detail in [26].

Here we generalise Pinkham's construction to exceptional curves with other normal bundles. Let  $\overline{H}$  be a normal surface singularity (in the end  $H$  will be a general hyperplane section of a 3-fold singularity) and let  $\hat{H}$  be a partial resolution of  $\overline{H}$  with irreducible exceptional locus  $C$ , such that the only singularities of  $\hat{H}$  are hypersurface singularities.

The deformation space of  $\hat{H}$  is smooth. Indeed, the sheaf  $\mathcal{T}_{\hat{H}}^1$  is concentrated in the singular points, and  $\mathcal{T}_{\hat{H}}^2 = 0$ , as there are only hypersurface singularities. The local-to-global spectral sequence for  $T_{\hat{H}}^\bullet$  gives that  $T_{\hat{H}}^2 = 0$ , so all deformations are unobstructed, and moreover we get the exact sequence

$$0 \longrightarrow H^1(\mathcal{T}_{\hat{H}}^1) \longrightarrow T_{\hat{H}}^2 \longrightarrow \bigoplus_p T_{\hat{H},p}^1 \longrightarrow 0.$$

Therefore all singular points  $p \in \hat{H}$  can be smoothed independently.

We take  $\tilde{X}$  to be a 1-parameter smoothing of  $\hat{H}$  with smooth total space (this is possible, as all singularities are hypersurfaces). Moreover, we can arrange that the general fibre does not contain exceptional curves. Then the contraction  $\pi: \tilde{X} \rightarrow X$  with the curve  $C$  as exceptional locus gives an isolated 3-fold singularity. In general its hyperplane section  $H$  is a nonnormal surface singularity, with  $\delta(H) = p_g(\overline{H})$ , by Proposition 7.

**Example 6.** Let  $\overline{H}$  be a singularity, whose resolution has a central rational curve of self-intersection  $-n-1$ , intersected by  $2n+1$   $(-2)$ -curves. A quasi-homogeneous singularity with this resolution is the hypersurface singularity  $Y_{2n+1}$  with equation  $z^2 = f_{2n+1}(x, y)$ , where  $f_{2n+1}$  is a square-free binary form. The partial resolution  $\widehat{H}$  to be considered is obtained by blowing down the  $(-2)$ -curves, intersecting the central curve. For  $n > 1$  this is the canonical model, while for  $n = 1$  we have  $D_4$ . The next Theorem shows that the construction yields a 3-dimensional manifold  $\tilde{X}$  with an exceptional rational curve, whose normal bundle is  $\mathcal{O}(n) \oplus \mathcal{O}(-2n-1)$ .

Rational double points are absolutely isolated, i.e., they can be resolved by blowing up points. Each sequence of blowing ups gives a partial resolution. We define the *resolution depth* of an exceptional component  $E_i$  as the minimal number of blow ups required to obtain a partial resolution on which the curve  $E_i$  appears. This is the desingularisation depth of [17], shifted by one. It is easily computed from the resolution graph. The fact that  $C$  is smooth restricts the possible curves  $E_i$  in a rational double points configurations, which intersect  $C$ , to those with multiplicity one in the fundamental cycle of the configuration.

**Theorem 9.** *Let  $\tilde{X}$  be a 1-parameter smoothing with smooth total space of a partial resolution  $\widehat{H}$  of a normal surface singularity  $\tilde{H}$  with exceptional set a smooth rational curve  $C$  and  $k$  rational double points as singularities. Let  $-c$  be the self-intersection of the curve  $C$  on the minimal resolution  $\tilde{H}$  of  $\tilde{H}$ . Suppose that  $C$  intersects a curve of resolution depth  $b_j$  in the  $j$ th rational double point configuration on  $\tilde{H}$ . Put  $b = b_1 + \dots + b_k$ . Then the normal bundle of the exceptional curve  $C \subset \widehat{H}$  in  $\tilde{X}$  is  $\mathcal{O}(b-c) \oplus \mathcal{O}(-b)$ .*

*Proof.* Let  $\sigma: \tilde{Y} \rightarrow \tilde{X}$  be an embedded resolution of  $\widehat{H}$ . Denote by  $\tilde{H}$  the strict transform of  $\widehat{H}$  and by  $\tilde{C}$  the strict transform of the curve  $C$  (which is isomorphic to  $C$ ). As we are only interested in a neighbourhood of  $\tilde{C}$ , it actually suffices to blow up the threefold  $\tilde{X}$  in points lying on  $C$  until the strict transform of  $\widehat{H}$  is smooth along the strict transform of  $C$ . The number of blow ups needed is  $b$ .

Let  $P_j \in C$  be the  $j$ th singular point of  $\widehat{H}$ ; identifying  $\tilde{C}$  with  $C$  it is also the intersection point on  $\tilde{H}$  of  $\tilde{C}$  and the  $j$ th rational double point configuration. The normal bundle  $N_{\tilde{C}/\tilde{Y}}$  is isomorphic to  $N_{C/\tilde{X}} \otimes \mathcal{O}_C(-D)$ , where we write  $D$  for the divisor  $\sum b_j P_j$ .

On  $\tilde{Y}$  we have the exact sequence

$$0 \longrightarrow N_{\tilde{C}/\tilde{H}} \longrightarrow N_{\tilde{C}/\tilde{Y}} \longrightarrow N_{\tilde{H}/\tilde{Y}}|_{\tilde{C}} \longrightarrow 0.$$

Correspondingly there is an exact sequence on  $\tilde{X}$ :

$$(9) \quad 0 \longrightarrow N' \longrightarrow N_{C/\tilde{X}} \longrightarrow N'' \longrightarrow 0,$$

with  $N' \cong N_{\tilde{C}/\tilde{H}} \otimes \mathcal{O}_C(D)$  a bundle, which outside the singular points coincides with the normal bundle  $N_{C/\tilde{H}}$ ; note that  $C$  is not a Cartier divisor in  $\tilde{H}$  at the singular points.

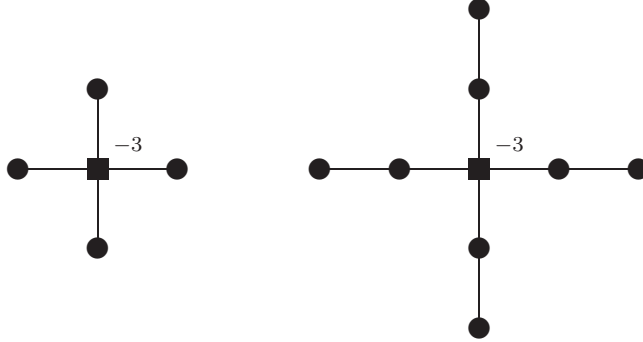
As  $\tilde{C}$  is a rational curve,  $N_{\tilde{C}/\tilde{H}} \cong \mathcal{O}_{\tilde{C}}(-c)$ . To compute  $N_{\tilde{H}/\tilde{Y}}|_{\tilde{C}}$  we note that the total transform of  $\widehat{H}$  is of the form  $\tilde{H} + \sum f_i F_i$ , with  $F_i$  the exceptional divisors,

and that it is the divisor  $t = 0$  with trivial normal bundle. The only divisors, which intersect  $\tilde{C}$ , are the ones coming from the last blow up in the points  $P_j$ , and occur with multiplicity  $2b_j$ . Therefore  $N_{\tilde{H}/\tilde{Y}}|_{\tilde{C}} \cong \mathcal{O}_{\tilde{C}}(-\sum 2f_j F_j) = \mathcal{O}_{\tilde{C}}(-2b)$ . It follows that the exact sequence (9) has the form

$$0 \longrightarrow \mathcal{O}(b - c) \longrightarrow N_{C/\tilde{X}} \longrightarrow \mathcal{O}(-b) \longrightarrow 0.$$

As  $H^0(N_{C/\tilde{X}}) = H^0(\mathcal{O}(b - c))$ , the sequence splits.  $\square$

**Example 7.** As noted by Ando [3], there exist exceptional rational curves with normal bundle  $(1, -4)$ , which contract to Cohen-Macaulay singularities, and others which do not. We obtain this normal bundle starting from a RDP resolution with a central rational curve of self-intersection  $-3$  (on the minimal resolution) with four  $A_i$  singularities on it. Consider the following two resolution graphs:



The graph on the left is a rational quadruple point graph, while a normal singularity with the second graph is minimally elliptic and equisingular to  $z^3 = x^4 + y^4$ . The exceptional  $(1, -4)$ -curve comes from a nonnormal model with  $\delta = 1$ .

**Example 8** (Example 6 continued). Ando's examples [1, 3] of the extremal case  $(n, -2n - 1)$  are of type in the Example. With adapted variable names his exceptional  $\mathbb{P}^1$  is covered by two charts having coordinates  $(x, \eta, \zeta_x)$  and  $(\xi, y, \zeta_y)$  with transition functions

$$\begin{aligned} x &= \xi^{2n+1}y + \zeta_y^2 + \xi^{2n}\zeta_y^3 \\ \eta &= \xi^{-1} \\ \zeta_x &= \zeta_y\xi^{-n} \end{aligned}$$

So  $x$  is a global function, as is  $\xi y + \zeta_y^3 = \eta^{2n}x - \zeta_x^2$ . Other functions are more complicated. A general hyperplane section is obtained by setting a linear combination of these two functions to zero. In the first chart we get  $\zeta_x^2 = x(a + \eta^{2n})$  and in the second  $\zeta_y^2 + (1/a + \xi^{2n})(\xi y + \zeta_y^3) = 0$ . This is indeed the canonical model of a singularity of type  $z^2 = f_{2n+1}(x, y)$ .

**Remark 3.** For  $n = 2$  we have the singularity  $z^2 = f_5$ , which is minimally elliptic and therefore every singularity with the same resolution graph is a double point. For  $n > 2$  this is no longer true. For  $n = 3$  we gave in Example 5 equations for a singularity with the same resolution graph as  $z^2 = f_7$ , which is not Gorenstein. As we constructed it as deformation of a nonnormal model, with  $\delta = 1$ , of a hypersurface singularity, a nonnormal  $\delta = 2$  model of this singularity is a deformation of a  $\delta = 3$  model of the hypersurface.

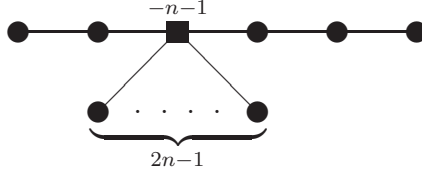
Before giving other new examples with normal bundle  $(n, -2n - 1)$  we recall Kollár's length invariant [7, Lecture 16]:

**Definition 4.** The length  $l$  of the small contraction  $\pi : (\tilde{X}, C) \rightarrow (X, p)$  with irreducible exceptional curve  $C$  is

$$l = \lg \mathcal{O}_{\tilde{X}} / \pi^* \mathfrak{m}_{X,p} .$$

The length is equal to the multiplicity of the maximal ideal cycle of  $\overline{H}$  at the strict transform of the exceptional curve  $H$ .

**Example 9.** Consider the graph:



An example of a normal surface singularity with this graph is

$$z^2 = y(y^{4n-2} + x^{6n-3}) .$$

These singularities can be thought of as generalisations of  $E_7$ , just as those of the type  $z^2 = f_{2n+1}(x, y)$  are generalisations of  $D_4$ . A smooth total space of a 1-parameter smoothing of the canonical model has as exceptional set a rational curve with normal bundle  $(n, -2n - 1)$ . The invariant  $l$  has the value 4.

For  $n = 2$  the singularity has  $p_g = 5$ .

From our computations in Section 2 we can draw the following conclusion.

**Proposition 10.** *The singularity obtained by contracting a rational curve with normal bundle  $(2, -5)$  has embedding dimension at least 7.*

Explicit equations for the case that the normalisation  $\overline{H}$  is given by  $z^2 = y^5 + x^5$ , can be obtained from the equations  $G_i, H_j$  of Section 2 by putting  $b = 1, a_4 = a_4 - y, a_3 = a_3 + b_4y, a_2 = a_2 + b_3y, a_1 = a_1 + b_2y + c_2y^2$  and  $a_0 = a_0$ .

The equation

$$H_1 = z^2 - (y - a_4)(y^2 + a_0t^2)^2 - x_3x_2 + ((a_3 + b_4y)v + (a_2 + b_3y)x_2)(y^2 + a_0t^2) \\ + (a_1 + b_2y + c_3y^2)x_2v + a_0x_2^2 + (a_3 + b_4y)t^3z + (y - a_4)t^4x_2 .$$

shows that restricted to  $t = 0$  one has the simultaneous normalisation

$$z^2 - y^5 - x^5 \\ + a_4y^4 + a_3xy^3 + a_2x^2y^2 + a_1x^3y + a_0x^4 + b_4xy^4 + b_3x^2y^3 + b_2x^3y^2 + c_3x^3y^3 .$$

To compute the simultaneous canonical model we first eliminate  $w$  and  $x_3$ , assuming  $t \neq 0$ . To simplify the formulas we suppress the  $b_i$  and  $c_3$ . We write the resulting equations in determinantal form:

$$(10) \quad \begin{pmatrix} v(y^2 + a_0t^2) - x_2yt^2 - zt^3 - a_3t^6 & v^2 - (y - a_4)t^6 \\ -(y^2 + a_0t^2)^2 + x_2t^4 & -v(y^2 + a_0t^2) - x_2yt^2 - zt^3 \\ 2y(y^2 + a_0t^2) - vt^2 - a_1t^4 & 2vy + x_2t^2 + a_2t^4 \end{pmatrix} .$$

We apply a Tjurina modification followed by normalisation. On the first chart  $\mathcal{U}_y$  we have the hypersurface

$$\zeta_y^2 - y\xi^5 - y + a_0\xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 - \frac{1}{4}t^2\xi^6$$

and the map to the singularity  $H$  is given by quite complicated formulas:

$$\begin{aligned} x_2 &= y^2\xi^2 - 2yt(\zeta_y - t\xi^3) + -t^3\xi(\zeta_y - \frac{1}{2}t\xi^3) - a_0t^2\xi^2 - a_1t^2\xi - a_2t^2 \\ v &= (y^2 + a_0t^2)\xi - yt^2\xi^2 + t^3(\zeta_y - \frac{1}{2}t\xi^3) \\ z &= y^2(\zeta_y - \frac{5}{2}t\xi^3) + yt^2\xi(3\zeta_y - \frac{5}{2}t\xi^3) + t^4\xi^2(\zeta_y - \frac{1}{2}t\xi^3) \\ &\quad - a_0t^2(\zeta_y - \frac{3}{2}t\xi^3) + 2ya_0t\xi^2 + ya_1t\xi + a_1t^3\xi^2 + ya_2t + a_2t^3\xi \end{aligned}$$

The expressions for  $w$  and  $x_3$  are even longer; they can be computed from the equations used to eliminate these variables.

In the second chart  $\mathcal{U}_x$  we have the hypersurface

$$\zeta_x^2 - x - x\eta^5 + a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 - \frac{1}{4}t^2\eta^8$$

and the transition functions  $y = x\eta + t\zeta_x + \frac{1}{2}t^2\eta^4$ ,  $\xi = \eta^{-1}$  and  $\zeta_y = \zeta_x\eta^{-2} + \frac{1}{2}t\eta^{-3} + \frac{1}{2}t\eta^2$ .

We see that for  $a_i = 0$  the family of curves given in  $\mathcal{U}_y$  by  $y = \zeta_y - \frac{1}{2}t\xi^3 = 0$  and in  $\mathcal{U}_x$  by  $x = \zeta_x + \frac{1}{2}t\eta^4 = 0$  is exceptional. Then the determinant (10) describes for  $t \neq 0$  a singularity isomorphic to the cone over the rational normal curve of degree three. The general 1-parameter smoothing of the canonical model is obtained by taking the  $a_i$  as functions of  $t$ . We can take  $a_0 = t$ ,  $a_i = 0$  for  $i > 0$ . Then in the second chart  $\tau = t(1 - \frac{1}{4}t\eta^8)$  can be eliminated, as  $\tau = x(1 + \eta^5) - \zeta_y^2$ , so  $(x, \eta, \zeta_y)$  are coordinates. We can write the equation in the first chart as  $\zeta_y^2 + \xi^4(t - \frac{1}{4}t\xi^2 - y\xi) = y$ , so  $(\zeta_y, \xi, \sigma = t - \frac{1}{4}t\xi^2 - y\xi)$  are coordinates. The transition functions are power series.

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